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Rendezvous Problem for Nearly Circular Orbits

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In order to assess various closure techniques during the terminal phase of a rendezvous maneuver, it is desirable to know the free flight motion of one vehicle relative to another. When expressed in terms of a reference system centered at the target vehicle, the equations of motion that are to be solved are a set of nonlinear differential equations with explicit time-dependent coefficients. In this paper an approximate analytical solution is determined by utilizing perturbation techniques. The solution is more general than those previously found since it includes the case where the target orbit possesses small eccentricity, and it remains valid for relatively large distances between the vehicles. Hence the results are useful for many near-earth missions. Numerical examples are provided in order to assess the accuracy of the solution. The results indicate that in certain instances the present solution, which includes some non-linear effects, must be used in lieu of previous solutions that are based on a linear theory. An approximate analytical solution is also obtained by which the required velocity components can be determined in order to effect rendezvous at a prescribed time.

Nomenclature

a_T, e, M = semimajor axis, eccentricity, and mean anomaly of target orbit
$\bar{R}$ = relative position vector to interceptor $(\bar{R}_I - \bar{R}_T)$
$\bar{R}_I, \bar{R}_T$ = inertial position vectors to interceptor and target vehicles
t, t_p = time, and time of periapsis passage
$\bar{V}$ = relative velocity of interceptor
x, y, z = nondimensional coordinates
X, Y, Z = coordinates of interceptor relative to target
$\delta z, \delta y, \delta z$ = variations of nondimensional coordinates from Clohessy-Wiltshire results
θ, ω = angular coordinate and angular speed of target vehicle
μ = gravitational constant
ρ = R_T/a_T
τ = $(\mu/a_T{}^3)^{1/2}t$
τ^* = specified nondimensional time to rendezvous
φ = angle denoting direction of initial velocity difference, measured from the X axis

Subscripts

c = associated with solution to linearized equations of motion, target in circular path
0 = denotes conditions at $t = 0$

Superscripts

$(\dot{\ }), (\ )'$ = derivatives with respect to t and τ, respectively
$0, e$ = indices used in identifying coefficients in the solution

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I. Introduction

BECAUSE the rendezvous maneuver is an important part of space operations, several studies have been conducted on its various phases, from the launch phase to the final phase where docking is imminent. This paper is concerned with an analytical investigation of the terminal phase of the rendezvous maneuver prior to docking.

For the purpose of determining the advantages or disadvantages of different rendezvous techniques, it is essential to know the free flight motion of one vehicle relative to the other. Conceptually, this motion can be readily determined if the vehicles are treated as point masses in a Newtonian force field since the motion of each vehicle is then Keplerian. Unfortunately, however, the respective motions are given indirectly as functions of time and consequently a quantitative determination of the relative positions and velocities is difficult and may lead to numerical inaccuracies, particularly in the phase just prior to docking. For these reasons it is advantageous to use the relative equations of motion and to express the coordinates explicitly in terms of time.

In general, the system of differential equations that describe the motion of one vehicle relative to another is nonlinear and explicitly time dependent. In view of the complexity of the equations, exact analytic solutions are not obtainable. Nevertheless, certain approximate analytic solutions that are of practical utility have been previously extracted. The case of circular target orbits has been investigated by Clohessy and Wiltshire[1] who took into account linear terms in the relative distance, arising in the differential equations, and also by London[2] who accounted for quadratic terms, thereby extending the validity of the solution over larger relative distances. In addition, de Vries[3] has considered target orbits with small eccentricities, and by retaining only linear terms in the relative distances, has obtained an approximate solution to the equations through the use of a perturbation technique, where the independent variable was selected to be the true anomaly of the target orbit.

This paper obtains an approximate solution of the equations of relative motion, including linear and quadratic terms in the relative distance, for the case of nearly circular target orbits. The motion of the target vehicle is approximated by means of power series in the eccentricity of its orbit. The new solution is obtained in terms of the independent variable, time, by the method of differential corrections accounting for terms of second order in relative distance and first order in target orbital eccentricity. The earlier results of[1] and[2] are recovered as special cases of this new solution.

II. Formulation and Analysis

Figure 1 indicates the target centered coordinate system that is used in this analysis. A rectangular (X, Y, Z) system is selected among the various possible coordinate systems that can be used, as described in[4] and.[5] The X and Y axes lie in the orbital plane with the Y axis directed radially and the X axis directed along the local horizontal. The Z axis completes a right-handed triad. Point O denotes the inverse-square force center, and $\dot{\theta}$ denotes the angular speed of the target vehicle.

In terms of an inertial system centered at 0, the two vehicles follow Keplerian motions described by

$$d^2\bar{R}_T/dt^2 = -\mu\bar{R}_T/R_T{}^3 \qquad d^2\bar{R}_I/dt^2 = -(\mu\bar{R}_I/R_I{}^3) \quad (1)$$

Since the position vector from the target to the interceptor is given by $\bar{R} = \bar{R}_I - \bar{R}_T$, the motion of the interceptor relative to the target is determined by

$$d^2\bar{R}/dt^2 = \mu[\bar{R}_T/R_T{}^3 - \bar{R}_I/R_I{}^3] \quad (2)$$

The acceleration is given in terms of the rotating system[6] by

$$d^2\bar{R}/dt^2 = \ddot{\bar{R}} + 2\bar{\omega}\mathbf{x}\dot{\bar{R}} + \bar{\omega}\mathbf{x}(\bar{\omega}\mathbf{x}\bar{R}) + \dot{\bar{\omega}}\mathbf{x}\bar{R} \quad (3)$$

where the dots denote time derivatives as viewed in the (X, Y, Z) system. If $(\bar{I}, \bar{J}, \bar{K})$ denote unit vectors along the X, Y, and Z axes, respectively, then the angular velocity is given by

$$\bar{\omega} = \dot{\theta}\bar{K} \quad (4)$$

The position vectors $\bar{R}_T$ and $\bar{R}$ are described by

$$\bar{R}_T = R_T\bar{J} \qquad \bar{R} = X\bar{I} + Y\bar{J} + Z\bar{K} \quad (5)$$

so that

$$\bar{R}_I = X\bar{I} + (Y + R_T)\bar{J} + Z\bar{K} \quad (6)$$

Using Eqs. (3–6), Eq. (2) yields the equation of motion of the interceptor vehicle in terms of the rotating coordinate system. The scalar form of this equation is given by

$$\left.\begin{aligned} &\ddot{X} - Y\ddot{\theta} - 2\dot{Y}\dot{\theta} - \dot{\theta}^2X + \mu X/[X^2 + \\ &\qquad (Y + R_T)^2 + Z^2]^{3/2} = 0 \\ &\ddot{Y} + X\ddot{\theta} + 2\dot{X}\dot{\theta} - \dot{\theta}^2Y - (\mu/ + R_T{}^2) \\ &\qquad \{\mu(Y + R_T)/[X^2 + (Y + R_T)^2 + Z^2]^{3/2}\} = 0 \\ &\ddot{Z} + \mu Z/[X^2 + (Y + R_T)^2 + Z^2]^{3/2} = 0 \end{aligned}\right\} \quad (7)$$

The gravitational terms bring about the nonlinear character of the equations. In addition, the coefficients $\dot{\theta}$ and $\ddot{\theta}$, as well as R_T, are time varying, unless the target orbit is circular.

It is convenient to introduce the nondimensional variables

$$\begin{aligned} x &= X/a_T \qquad y = Y/a_T \qquad z = Z/a_T \\ \rho &= R_T/a_T \qquad \tau = (\mu/a_T{}^3)^{1/2}t \end{aligned} \quad (8)$$

In terms of these variables, Eqs. (7) become

$$\left.\begin{aligned} &x'' - y\theta'' - 2y'\theta' - \theta'^2x + \\ &\qquad x/[x^2 + (y + \rho)^2 + z^2]^{3/2} = 0 \\ &y'' + x\theta'' + 2x'\,\theta' - \theta'^2y - (1/\rho^2) + \\ &\qquad \{(y + \rho)/[x^2 + (y + \rho)^2 + z^2]^{3/2}\} = 0 \\ &z'' + z/[x^2 + (y + \rho)^2 + z^2]^{3/2} = 0 \end{aligned}\right\} \quad (9)$$

where primes denote differentiation with respect to τ. It is now assumed that the distance between the two vehicles is small compared to the semimajor axis of the target vehicle

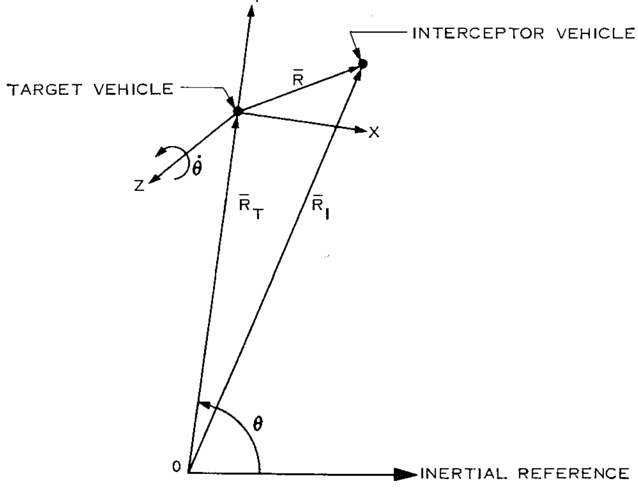

Fig. 1 Coordinate system.

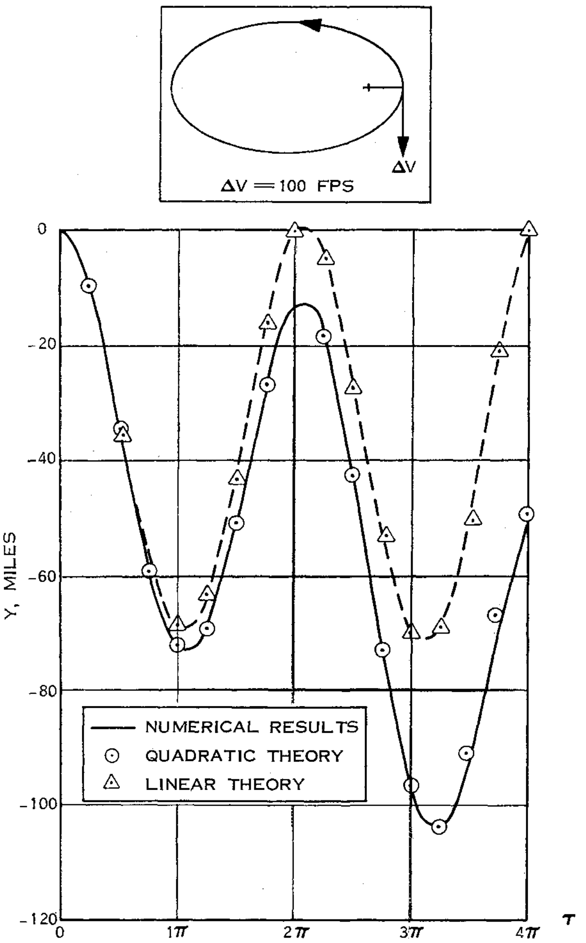

Fig. 2 Y, case I.

orbit. A useful approximation to the differential equations can be obtained by expanding the nonlinear terms in powers of the coordinates, retaining the linear and quadratic terms, which leads to

$$\left.\begin{aligned} &x'' - y\theta'' - 2y'\theta' + [(1/\rho^3) - \theta'^2]x - (3xy/\rho^4) = 0 \\ &y'' + x\theta'' + 2x'\theta' - [(2/\rho^3) + \theta'^2]y - \\ &\qquad (3/2\rho^4)/(x^2 - 2y^2 + z^2) = 0 \\ &z'' + z/\rho^3 - 3zy/\rho^4 = 0 \end{aligned}\right\} \quad (10)$$

For orbits that are nearly circular, the dependence of the angular speed and the radius of the target orbit upon the independent variable, time, can be taken into account through use of series expansions in the eccentricity[7]:

$$\left.\begin{aligned} \dot{\theta} &= (\mu/a_T^3)^{1/2}[1 + 2e\cos M + \tfrac{5}{2}e^2\cos 2M + \ldots] \\ R_T &= a_T[1 - e\cos M + (e^2/2)(1 - \cos 2M) + \ldots] \end{aligned}\right\} \quad (11)$$

where the mean anomaly is given by $M = (\mu/a_T^3)^{1/2}(t - t_p)$, where t_p denotes the time of periapsis passage. In nondimensional form, these become

$$\left.\begin{aligned} \theta' &= 1 + 2e\cos(\tau - \tau_p) + \tfrac{5}{2}e^2\cos 2(\tau - \tau_p) + \ldots \\ \rho &= 1 - e\cos(\tau - \tau_p) + (e^2/2)[1 - \\ &\qquad \cos 2(\tau - \tau_p)] + \ldots \end{aligned}\right\} \quad (12)$$

If the nonlinear terms are omitted and the target orbit is circular, Eqs. (10) revert to the Clohessy-Wiltshire equations,

$$\left.\begin{aligned} &x_c'' - 2y_c' = 0 \\ &y_c'' + 2x_c' - 3y_c = 0 \\ &z_c'' + z_c = 0 \end{aligned}\right\} \quad (13)$$

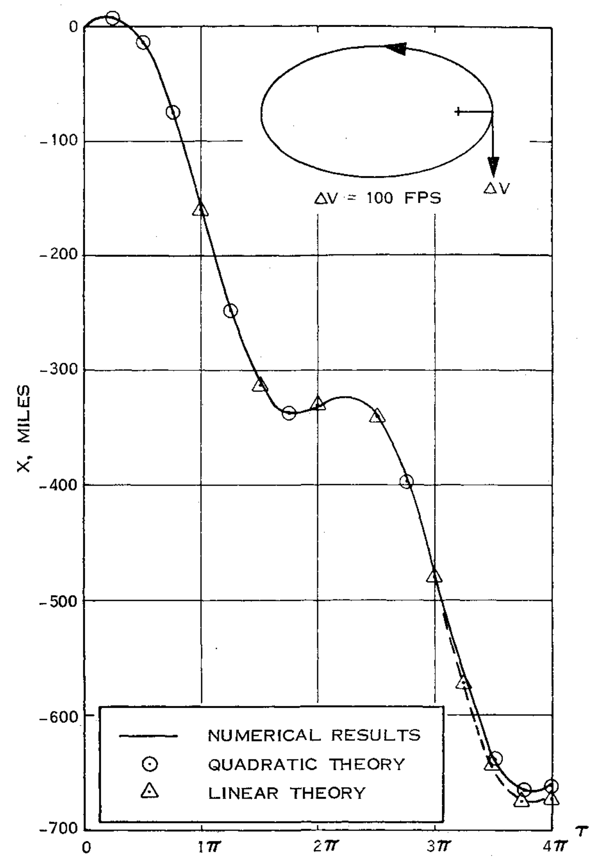

Fig. 3 X, case I.

where the subscript c denotes the linearized, circular orbit equations. For general initial conditions,

$$\left.\begin{aligned} x_c(0) &= x_0 & y_c(0) &= y_0 & z_c(0) &= z_0 \\ x_c'(0) &= x_0' & y_c'(0) &= y_0' & z_c'(0) &= z_0' \end{aligned}\right\} \quad (14)$$

the solution of Eqs. (13) is:

$$\left.\begin{aligned} x_c &= 2(2x_0' - 3y_0)\sin\tau - 2y_0'\cos\tau + \\ &\qquad 3(2y_0 - x_0')\tau + (x_0 + 2y_0') \\ y_c &= y_0'\sin\tau + (2x_0' - 3y_0)\cos\tau + 2(2y_0 - x_0') \\ z_c &= z_0'\sin\tau + z_0\cos\tau \end{aligned}\right\} \quad (15)$$

In order to improve on this solution, we use the method of differential corrections. By defining δx, δy, and δz by

$$x = x_c + \delta x \qquad y = y_c + \delta y \qquad z = z_c + \delta z \quad (16)$$

and substituting into Eqs. (10), we obtain differential equation for the variations δx, δy, and δz. These are simplified by retaining only the larger quantities, such as quadratic terms in the coordinates (x_c^2, y_cz_c, etc.) and linear terms in the eccentricity (ex_c, etc.), and neglecting the smaller terms, such as $x_c\delta x$, $e\delta x$, ex_c^2, e^2y_c, etc. The resulting differential equations are

$$\left.\begin{aligned} &\delta x'' - 2\delta y' = 3x_cy_c + e[(4y_c' + x_c)\cos(\tau - \tau_p) - \\ &\qquad 2y_c\sin(\tau - \tau_p)] \\ &\delta y'' + 2\delta x' - 3\delta y = \tfrac{3}{2}(x_c^2 - 2y_c^2 + z_c^2) + \\ &\qquad e[(10y_c - 4x_c')\cos(\tau - \tau_p) + 2x_c\sin(\tau - \tau_p)] \\ &\delta z'' + \delta z = 3y_cz_c - 3e\,z_c\cos(\tau - \tau_p) \end{aligned}\right\} \quad (17)$$

The initial conditions for δx, $\delta x'$, etc. are all zero, because the general initial conditions have been satisfied by x_c, y_c, z_c. Since the variation equations are a set of linear equations with constant coefficients and known "forcing functions," the determination of the solution is straightforward but involves considerable effort. For convenience, the solution is carried out in two parts, indicated by

$$\delta x = \delta x^0 + e\delta x^e \qquad \delta y = \delta y^0 + e\delta y^e \qquad \delta z = \delta z^0 + e\delta z^e \tag{18}$$

where the superscript 0 denotes the solution when the target orbit is circular, while the quantities δx^e, δy^e, and δz^e reflect the first-order effect of eccentricity on the solution. These solutions are of the form

$$\left.\begin{aligned}
\delta x^p &= A_0{}^p + A_1{}^p \sin\tau + A_2{}^p \cos\tau + A_3{}^p \sin 2\tau + \\
&\qquad A_4{}^p \cos 2\tau + A_5{}^p\tau + A_6{}^p\tau \sin\tau + A_7{}^p\tau \cos\tau \\
\delta y^p &= B_0{}^p + B_1{}^p \sin\tau + B_2{}^p \cos\tau + B_3{}^p \sin 2\tau + \\
&\qquad B_4{}^p \cos 2\tau + B_5{}^p\tau + B_6{}^p\tau \sin\tau + \\
&\qquad B_7{}^p\tau \cos\tau + B_8{}^p \tau^2 \\
\delta z^p &= C_0{}^p + C_1{}^p \sin\tau + C_2{}^p \cos\tau + C_3{}^p \sin 2\tau + \\
&\qquad C_4{}^p \cos 2\tau + C_5{}^p\tau \sin\tau + C_6{}^p\tau \cos\tau
\end{aligned}\right\} \tag{19}$$

where, for convenience, the superscript p can denote either 0 or e and the $A_n{}^p$, $B_n{}^p$, $C_n{}^p$ are constants. The coefficients in the solutions δx^0, δy^0, and δz^0 are given by

$$\left.\begin{aligned}
A_0{}^0 &= 3(\tfrac{5}{2}y_0y_0' - x_0'y_0' + x_0y_0 + \tfrac{1}{2}z_0z_0') \\
A_1{}^0 &= -3x_0y_0' + 36\,x_0'y_0 - 10\,(x_0')^2 - \\
&\qquad 2(y_0')^2 - 30\,y_0^2 - 3x_0^2 - z_0^2 - 2(z_0')^2 \\
A_2{}^0 &= -6y_0y_0' + 2x_0'y_0' - 3x_0y_0 - 2z_0z_0' \\
A_3{}^0 &= -(x_0')^2 + \tfrac{1}{4}\,(y_0')^2 + 3x_0'y_0 - \tfrac{9}{4}y_0^2 + \\
&\qquad \tfrac{1}{4}(z_0')^2 - \tfrac{1}{4}z_0^2 \\
A_4{}^0 &= x_0'y_0' - \tfrac{3}{2}y_0y_0' + \tfrac{1}{2}z_0z_0' \\
A_5{}^0 &= 3[x_0^2 + \tfrac{11}{2}y_0^2 + 2(x_0')^2 + \tfrac{1}{2}(y_0')^2 + \\
&\qquad \tfrac{1}{2}\,(z_0')^2 + \tfrac{1}{2}z_0^2] - 21x_0'y_0 + 2x_0y_0' \\
A_6{}^0 &= -6y_0y_0' + 3x_0'y_0' \\
A_7{}^0 &= -21x_0'y_0 + 6(x_0')^2 + 18y_0^2 \\
B_0{}^0 &= \tfrac{21}{2}\,y_0^2 - \tfrac{3}{2}(y_0')^2 + \tfrac{3}{2}x_0^2 + 3(x_0')^2 - \\
&\qquad 12x_0'y_0 + \tfrac{3}{4}(z_0')^2 + \tfrac{3}{4}z_0^2 \\
B_1{}^0 &= 12y_0y_0' - 7x_0'y_0' - 3x_0x_0' + 6x_0y_0 + z_0z_0' \\
B_2{}^0 &= 2(y_0')^2 + 18x_0'y_0 - 5(x_0')^2 - 15y_0^2 - \\
&\qquad \tfrac{3}{2}x_0^2 - \tfrac{1}{2}z_0^2 - (z_0')^2 \\
B_3{}^0 &= 2x_0'y_0' - 3y_0y_0' - \tfrac{1}{2}z_0z_0' \\
B_4{}^0 &= -\tfrac{1}{2}(y_0')^2 + 2(x_0')^2 - 6x_0'y_0 + \tfrac{9}{2}y_0^2 + \\
&\qquad \tfrac{1}{4}\,(z_0')^2 - \tfrac{1}{4}z_0^2 \\
B_5{}^0 &= -6x_0y_0 + 3x_0x_0' + 6x_0'y_0' - 12y_0y_0' \\
B_6{}^0 &= -21x_0'y_0 + 6(x_0')^2 + 18y_0^2 \\
B_7{}^0 &= 6y_0y_0' - 3x_0'y_0' \\
B_8{}^0 &= -18y_0^2 + 18x_0'y_0 - \tfrac{9}{2}\,(x_0')^2 \\
C_0{}^0 &= \tfrac{3}{2}\,(y_0'z_0' + 2x_0'z_0 - 3y_0z_0) \\
C_1{}^0 &= y_0'z_0 - x_0'z_0' + 3y_0z_0' \\
C_2{}^0 &= -2y_0'z_0' - 2x_0'z_0 + 3y_0z_0 \\
C_3{}^0 &= -\tfrac{1}{2}\,(y_0'z_0 + 2x_0'z_0' - 3y_0z_0') \\
C_4{}^0 &= -\tfrac{1}{2}\,(2x_0'z_0 - 3y_0z_0 - y_0'z_0') \\
C_5{}^0 &= \tfrac{3}{2}\,(4y_0z_0 - 2x_0'z_0) \\
C_6{}^0 &= -\tfrac{3}{2}\,(4y_0z_0' - 2x_0'z_0')
\end{aligned}\right\} \tag{20}$$

This solution, which represents the effects of retaining quadratic terms in the relative distance, is essentially equivalent to that found by London.[2] Two discrepancies are observed, however, which involve the coefficients $A_5{}^0$ and $C_2{}^0$, which differ from London's results by the quantities $(-21\,x_0'y_0 + 3x_0y_0')$ and $(-\tfrac{1}{2}y_0'z_0' + \tfrac{1}{2}y_0z_0')$, respectively.

The coefficients in the expressions for δx^e, δy^e, and δz^e are given by the following:

$$\left.\begin{aligned}
A_0{}^e &= (-3x_0' + \tfrac{7}{2}y_0)\sin\tau_p + (x_0 - \tfrac{1}{2}y_0')\cos\tau_p \\
A_1{}^e &= 3(x_0 + 2y_0')\sin\tau_p - 12y_0\cos\tau_p \\
A_2{}^e &= (6x_0' - 8y_0)\sin\tau_p + (-x_0 + 2y_0')\cos\tau_p \\
A_3{}^e &= -\tfrac{3}{2}y_0'\sin\tau_p + \tfrac{3}{2}\,(2x_0' - 3y_0)\cos\tau_p \\
A_4{}^e &= \tfrac{3}{2}\,(-2x_0' + 3y_0)\sin\tau_p - \tfrac{3}{2}y_0'\cos\tau_p \\
A_5{}^e &= -(3x_0 + 3y_0')\sin\tau_p + (-3x_0' + 15y_0)\cos\tau_p \\
A_6{}^e &= -3\,(x_0' - 2y_0)\sin\tau_p \\
A_7{}^e &= -3(x_0' - 2y_0)\cos\tau_p \\
B_0{}^e &= -(2x_0 + 3y_0')\sin\tau_p + (-4x_0' + 13y_0)\cos\tau_p \\
B_1{}^e &= -x_0'\sin\tau_p - 2y_0'\cos\tau_p \\
B_2{}^e &= 2(x_0 + 2y_0')\sin\tau_p + (2x_0' - 10y_0)\cos\tau_p \\
B_3{}^e &= (2x_0' - 3y_0)\sin\tau_p + y_0'\cos\tau_p \\
B_4{}^e &= -y_0'\sin\tau_p + (2x_0' - 3y_0)\cos\tau_p \\
B_5{}^e &= 0 \\
B_6{}^e &= (3x_0' - 6y_0)\cos\tau_p \\
B_7{}^e &= (-3x_0' + 6y_0)\sin\tau_p \\
B_8{}^e &= 0 \\
C_0{}^e &= -\tfrac{3}{2}z_0'\sin\tau_p - \tfrac{3}{2}z_0\cos\tau_p \\
C_1{}^e &= -z_0\sin\tau_p - z_0'\cos\tau_p \\
C_2{}^e &= 2z_0'\sin\tau_p + z_0\cos\tau_p \\
C_3{}^e &= \tfrac{1}{2}z_0\sin\tau_p + \tfrac{1}{2}z_0'\cos\tau_p \\
C_4{}^e &= -\tfrac{1}{2}z_0'\sin\tau_p + \tfrac{1}{2}z_0\cos\tau_p \\
C_5{}^e &= 0 \\
C_6{}^e &= 0
\end{aligned}\right\} \tag{21}$$

To summarize, an improved approximate solution of the equations of motion has been obtained which is valid for nearly circular target orbits. The new solution is obtained by adding to the coordinates of the Clohessy-Wiltshire solution, given by Eqs. (15), the variations in coordinates given by Eqs. (18–21).

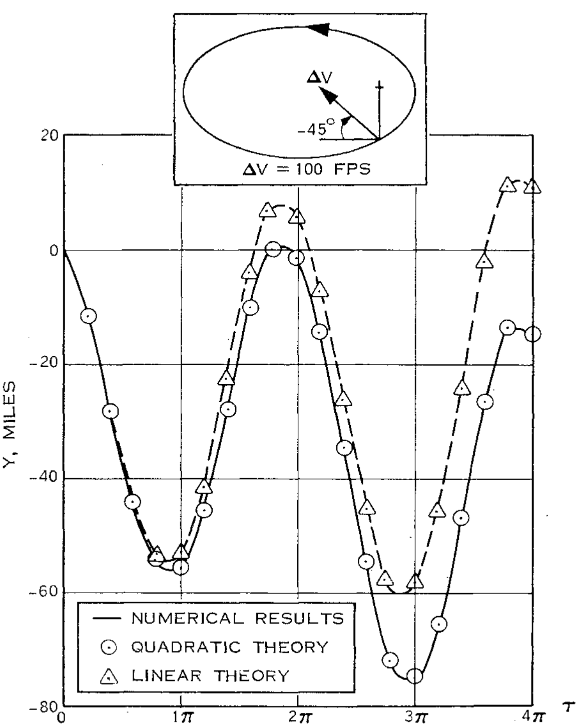

Fig. 4 Y, case II.

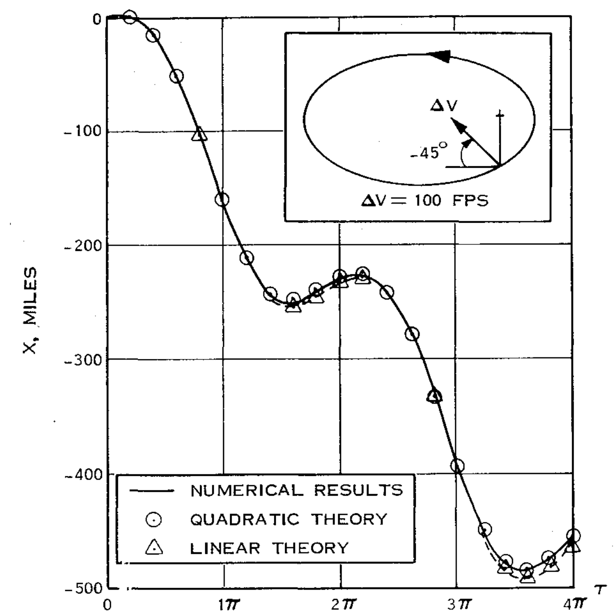

Fig. 5 X, case II.

In order to evaluate the accuracy of the new solution, some comparisons have been made between it and independent results obtained numerically via Kepler's equation. The calculations were simplified by taking initial conditions so that the interceptor would move in the orbital plane of the target vehicle, i.e., $z \equiv 0$. Also, for convenience, the calculations were carried out backward in time, i.e., the vehicles were assumed to initially occupy the same position but to have different velocities.

Table 1. Initial positions and velocity differences for accuracy evaluations

Case	τ-	ΔV, fps	φ
I	0	100	0
II	$\pi/2$	100	$-\pi/4$
III	$\pi/2$	400	$-\pi/4$
IV	π	827.52	π

In the four cases presented, perigee and apogee of the target orbit were chosen to be 200 and 400 miles above the surface of the earth, respectively ($e = 0.0234$). For each case the initial position τ_p and the velocity difference ΔV are listed in Table 1. In addition, the pulse direction is denoted by the angle φ, which is measured positive in a counter-clockwise sense from the positive X direction. The numerical results for cases I, II, and III are shown in Figs. 2–7. For comparison purposes, results based on linear theory are shown also. To obtain these, the eccentricity of the target orbit is taken into account but quadratic terms in the relative distance are neglected. As expected, the linearized predictions are accurate for a short time until the distances involved become rather large, which occurs in approximately $\frac{1}{2}$ revolution ($\tau = \pi$) of the target orbit for the cases investigated. Thereafter, the linearized results differ substantially from the numerical solutions, particularly in the vertical component of

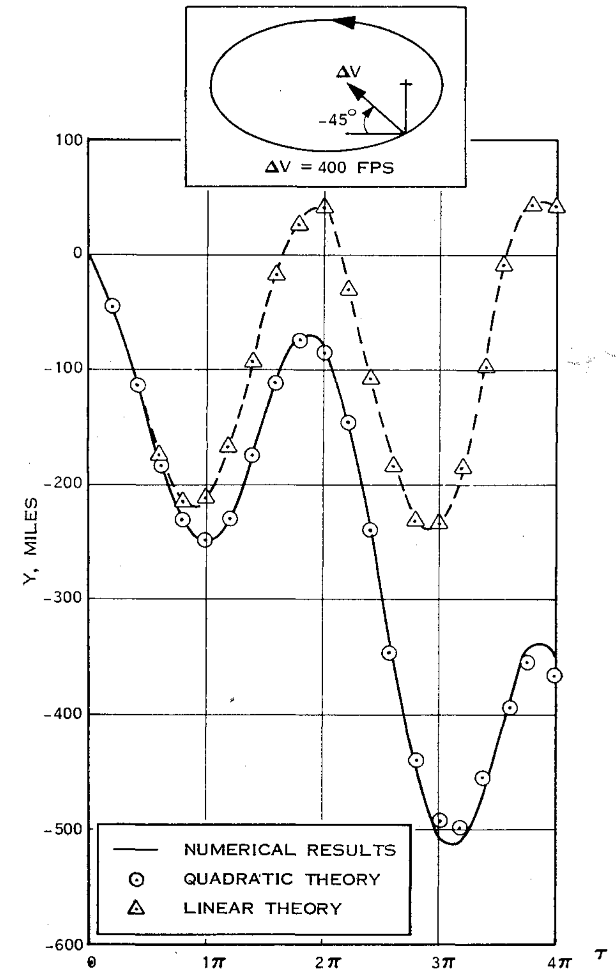

Fig. 6 Y, case III.

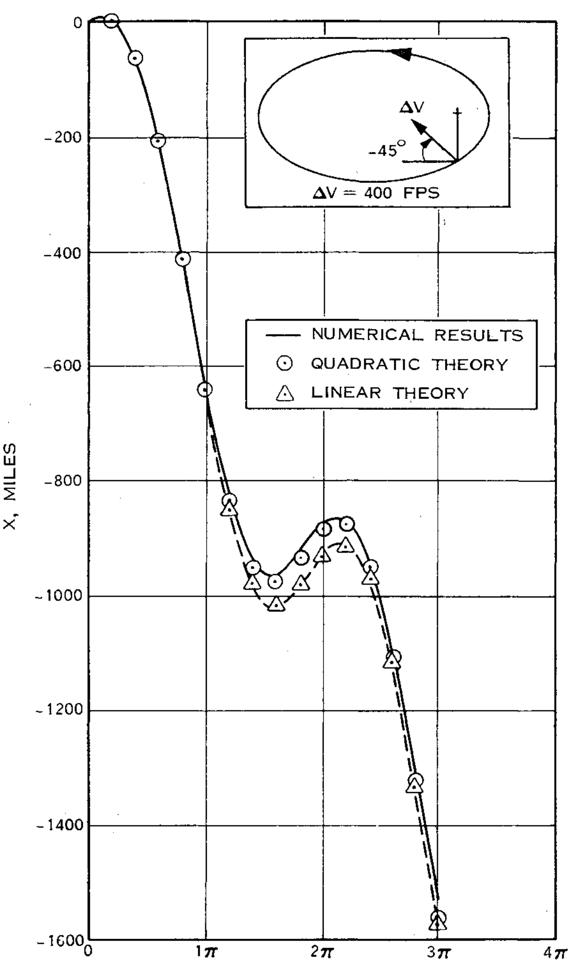

Fig. 7 X, case III.

the distance, while the present approximate analysis yields accurate results for about 2 revolutions of the target.

Figures 8 and 9 indicate the accuracy of the present analysis for a rather extreme case of a large initial velocity difference ($\Delta V = 827.52$ fps, which yields an apogee altitude of 800 miles for the interceptor) in which rapid breakdown of the predictions can be expected. Even for this extreme case, the qualitative features of both X and Y are predicted by the quadratic solution, and for relative distances less than 2500 miles, the predictions are quantitatively accurate.

III. Velocity Components for Rendezvous

We shall now apply the results found in the previous section to the problem of determining the relative velocity required of an interceptor so that it will rendezvous with the target at a prescribed nondimensional time τ^*. Such a rendezvous requires that

$$x(\tau^*) = y(\tau^*) = z(\tau^*) = 0 \tag{22}$$

which constitute three equations to determine the unknown relative velocity components x_0', y_0', z_0' in terms of the known quantities x_0, y_0, z_0, and τ^*. Each of Eqs. (22), in its dependence on the velocity components, is of the form

$$(\text{linear terms}) + (\text{quadratic terms}) + f(x_0, y_0, z_0, \tau^*) = 0 \tag{23}$$

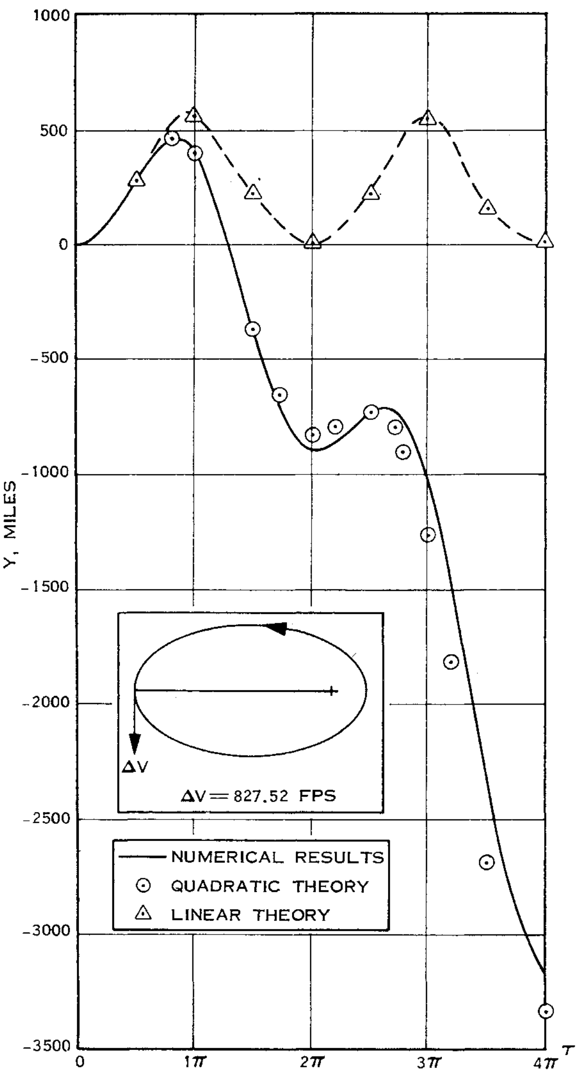

Fig. 8 ***Y*, case IV.**

For any specific case, these equations may be solved numerically by an iterative procedure. This would yield results of similar accuracy to those of the previous section. However, with an additional assumption, it is possible to determine an approximate solution of Eqs. (23) in the form

$$x_0' = x_l' + \epsilon_x \qquad y_0' = y_l' + \epsilon_y \qquad z_0' = z_l' + \epsilon_z \tag{24}$$

where (x_l', y_l', z_l') is the solution of Eqs. (23) when the quadratic terms in the initial velocity components are omitted. By substituting Eqs. (24) into Eqs. (23) and assuming that the corrections $(\epsilon_x, \epsilon_y, \epsilon_z)$ are small enough that their squares and cross products are negligible, we find that the corrections satisfy a linear system of algebraic equations.

The equations to be solved for x_l', y_l', z_l' are

$$\left.\begin{aligned} L_0\, x_l' + L_1\, y_l' + L_2\, z_l' + L_9 &= 0 \\ M_0\, x_l' + M_1\, y_l' + M_2\, z_l' + M_9 &= 0 \\ N_0\, x_l' + N_1\, y_l' + N_2\, z_l' + N_9 &= 0 \end{aligned}\right\} \tag{25}$$

where

$$\left.\begin{aligned} L_0 = {} & 4(1 + 9y_0) \sin\tau^* + 3y_0 \sin 2\tau^* - 3(1 + 7y_0)\tau^* - 21y_0\tau^* \cos\tau^* + e[(-3 + 6 \cos\tau^* - 3 \cos 2\tau^* - 3\tau^* \sin\tau^*) \sin\tau_p + {} \\ & (-3\tau^* \cos\tau^* + 3 \sin 2\tau^* - 3\tau^*) \cos\tau_p] \\ L_1 = {} & \tfrac{1}{2}(4 + 15y_0) - 3x_0 \sin\tau^* - 2(1 + 3y_0) \cos\tau^* + 3x_0\tau^* - \tfrac{3}{2}y_0 \cos 2\tau^* - 6y_0\tau^* \sin\tau^* + e[(6 \sin\tau^* - \tfrac{3}{2} \sin 2\tau^* - {} \\ & 3\tau^*) \sin\tau_p + (-\tfrac{1}{2} + 2 \cos\tau^* - \tfrac{3}{2} \cos 2\tau^*) \cos\tau_p] \\ L_2 = {} & \tfrac{3}{2}z_0 - 2z_0 \cos\tau^* + \tfrac{1}{2}z_0 \cos 2\tau^* \\ L_9 = {} & x_0 + 3x_0y_0 + (-6y_0 - 30y_0^2 - 3x_0^2 - z_0^2) \sin\tau^* - 3x_0y_0 \cos\tau^* + (-\tfrac{9}{4}y_0^2 - \tfrac{1}{4}z_0^2) \sin 2\tau^* + {} \\ & 3(2y_0 + x_0^2 + \tfrac{11}{2} y_0^2 + \tfrac{1}{2}z_0^2)\tau^* + 18y_0^2\tau^* \cos\tau^* + e[(\tfrac{7}{2}y_0 + 3x_0 \sin\tau^* - 8y_0 \cos\tau^* + \tfrac{9}{2}y_0 \cos 2\tau^* - {} \\ & 3x_0\tau^* + 6y_0\tau^* \sin\tau^*) \sin\tau_p + (x_0 - 12y_0 \sin\tau^* - x_0 \cos\tau^* - \tfrac{9}{2}y_0 \sin 2\tau^* + 15y_0\tau^* + 6y_0\tau^* \cos\tau^*) \cos\tau_p] \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned} M_0 = {} & -2(1 + 6y_0) - 3x_0 \sin\tau^* + 2(1 + 9y_0) \cos\tau^* - 6y_0 \cos 2\tau^* + 3x_0\tau^* - 21y_0\tau^* \sin\tau^* + 18y_0\tau^{*2} + e[(-\sin\tau^* + {} \\ & 2 \sin 2\tau^* - 3\tau^* \cos\tau^*) \sin\tau_p + (-4 + 2 \cos\tau^* + 2 \cos 2\tau^* + 3\tau^* \sin\tau^*) \cos\tau_p] \\ M_1 = {} & (1 + 12y_0) \sin\tau^* - 3y_0 \sin 2\tau^* - 12y_0\tau^* + 6y_0\tau^* \cos\tau^* + e[(-3 + 4 \cos\tau^* - {} \\ & \cos 2\tau^*) \sin\tau_p + (-2 \sin\tau^* + \sin 2\tau^*) \cos\tau_p] \\ M_2 = {} & z_0 \sin\tau^* - \tfrac{1}{2}z_0 \sin 2\tau^* \\ M_9 = {} & 4y_0 + \tfrac{21}{2} y_0^2 + \tfrac{3}{2}x_0^2 + \tfrac{3}{4}z_0^2 + 6x_0y_0 \sin\tau^* + (-3y_0 - 15y_0^2 - \tfrac{3}{2}x_0^2 - \tfrac{1}{2}z_0^2) \cos\tau^* + (\tfrac{9}{2}y_0^2 - {} \\ & \tfrac{1}{4}z_0^2) \cos 2\tau^* - 6x_0y_0\tau^* + 18y_0^2\tau^* \sin\tau^* - 18y_0^2\tau^{*2} + e[(-2x_0 + 2x_0 \cos\tau^* - 3y_0 \sin 2\tau^* + {} \\ & 6y_0\tau^* \cos\tau^*) \sin\tau_p + (13y_0 - 10y_0 \cos\tau^* - 3y_0 \cos 2\tau^* - 6y_0\tau^* \sin\tau^*) \cos\tau_p] \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} N_0 &= 3z_0 - 2z_0 \cos\tau^* - z_0 \cos 2\tau^* - 3z_0\tau^* \sin\tau^* \\ N_1 &= z_0 \sin\tau^* - \tfrac{1}{2} z_0 \sin 2\tau^* \\ N_2 &= (1 + 3y_0) \sin\tau^* + \tfrac{3}{2} y_0 \sin 2\tau^* - \\ &\quad 6y_0\tau^* \cos\tau^* + e\,[(-\tfrac{3}{2} + 2\cos\tau^* - \\ &\quad \tfrac{1}{2}\cos 2\tau^*) \sin\tau_p + (-\sin\tau^* + \tfrac{1}{2}\sin 2\tau^*)\cos\tau_p] \\ N_9 &= -\tfrac{9}{2} y_0 z_0 + (z_0 + 3y_0 z_0)\cos\tau^* + \\ &\quad \tfrac{3}{2} y_0 z_0 \cos 2\tau^* + 6y_0 z_0 \tau^* \sin\tau^* + \\ &\quad e\,[(-z_0 \sin\tau^* + \tfrac{1}{2} z_0 \sin 2\tau^*)\sin\tau_p + \\ &\quad (-\tfrac{3}{2} z_0 + z_0 \cos\tau^* + \tfrac{1}{2} z_0 \cos 2\tau^*)\cos\tau_p] \end{aligned}\right\} \quad (28)$$

The equations to be solved for ϵ_x, ϵ_y, ϵ_z are

$$\left.\begin{aligned} Q_0\epsilon_x + Q_1\epsilon_y + Q_2\epsilon_z + Q_3 &= 0 \\ P_0\epsilon_x + P_1\epsilon_y + P_2\epsilon_z + P_3 &= 0 \\ S_0\epsilon_x + S_1\epsilon_y + S_2\epsilon_z + S_3 &= 0 \end{aligned}\right\} \quad (29)$$

where

$$\left.\begin{aligned} Q_0 &= L_0 + 2L_3 x_i' + L_6 y_i' + L_7 z_i' \\ Q_1 &= L_1 + 2L_4 y_i' + L_6 x_i' + L_8 z_i' \\ Q_2 &= L_2 + 2L_5 z_i' + L_7 x_i' + L_8 y_i' \\ Q_3 &= L_3 x_i'^2 + L_4 y_i'^2 + L_5 z_i'^2 + L_6 x_i' y_i' + \\ &\qquad L_7 x_i' z_i' + L_8 y_i' z_i' \end{aligned}\right\} \quad (30)$$

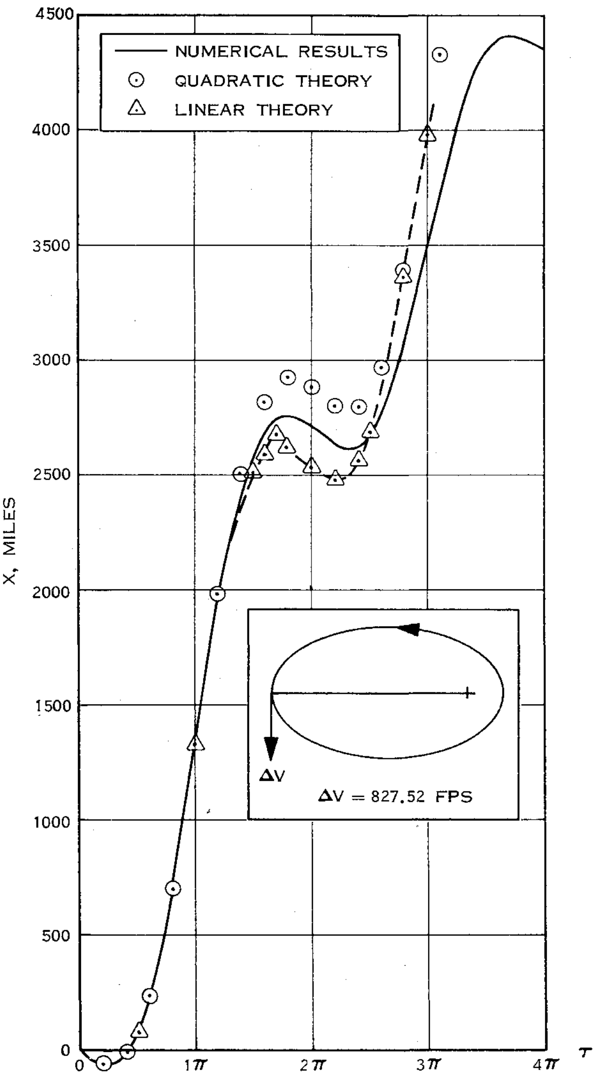

Fig. 9 X, case IV.

The corresponding expressions for $P_0, \ldots, P_3$ are obtained by replacing coefficients $L_0, \ldots, L_8$ in Eq. (30) by $M_0, \ldots, M_8$. Similarly, the expressions for $S_0, \ldots, S_3$ are the same as that in Eqs. (30) except that $L_0, \ldots, L_8$ are replaced by $N_0, \ldots, N_8$. The additional constants that are introduced are

$$\left.\begin{aligned} L_3 &= -10\sin\tau^* - \sin 2\tau^* + 6\tau^* + 6\tau^*\cos\tau^* \\ L_4 &= -2\sin\tau^* + \tfrac{1}{4}\sin 2\tau^* + \tfrac{3}{2}\tau^* \\ L_5 &= L_4 \\ L_6 &= -3 + 2\cos\tau^* + \cos 2\tau^* + 3\tau^*\sin\tau^* \\ L_7 &= L_8 = 0 \\ M_3 &= 3 - 5\cos\tau^* + 2\cos 2\tau^* + 6\tau^*\sin\tau^* - \tfrac{9}{2}\tau^{*2} \\ M_4 &= -\tfrac{3}{2} + 2\cos\tau^* - \tfrac{1}{2}\cos 2\tau^* \\ M_5 &= -\tfrac{1}{2}M_4 \\ M_6 &= -7\sin\tau^* + 2\sin 2\tau^* + 6\tau^* - 3\tau^*\cos\tau^* \\ M_7 &= M_8 = 0 \\ N_3 &= N_4 = N_5 = N_6 = 0 \\ N_7 &= -\sin\tau^* - \sin 2\tau^* + 3\tau^*\cos\tau^* \\ N_8 &= \tfrac{3}{2} - 2\cos\tau^* + \tfrac{1}{2}\cos 2\tau^* \end{aligned}\right\} \quad (31)$$

Thus, by solving the two sets of linear equations, Eqs. (25) and (29), by standard methods, approximate solutions for the initial velocity components may be found. The results found in this way are general in the sense that they are literal, but the region of their validity is restricted by the assumption that quadratic terms in the corrections are negligible. In order to assess the accuracy of these results, we construct two test cases in which the target orbit is the same as that for cases I–IV, and the interceptor moves in the plane of the target orbit.

For case V, we first determine (via Keplerian analysis, for specific times, such as $\tau = -0.1\,\pi$, $-0.2\,\pi$, etc.) the relative position and velocity of a vehicle which at $\tau = 0$ is coincident

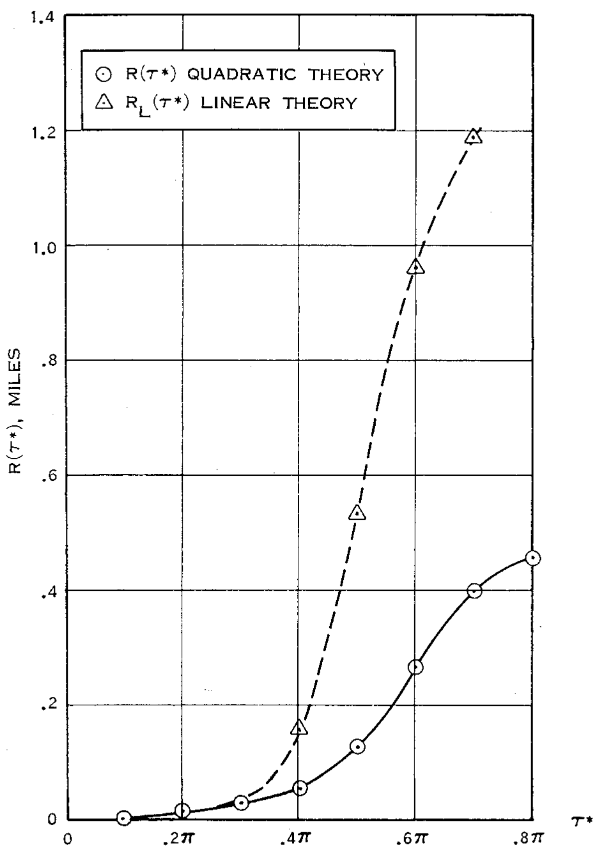

Fig. 10 Miss distance, case V.

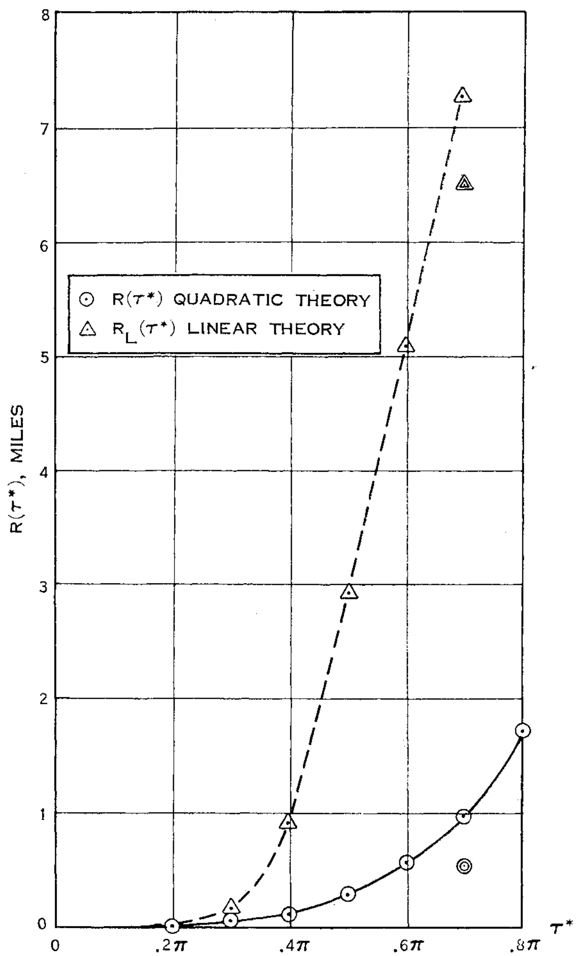

Fig. 11 Miss distance, case VI.

with the target vehicle at its perigee, and moves in a direction tangent to the target orbit with a speed of 100 fps larger than the target speed. Then by using the relative coordinates of the vehicle, at say $\tau = -0.2\ \pi$, we find the velocity components required to rendezvous at $\tau^* = 0.2\ \pi$ by the approximate method. Although the velocity components found in this way can be compared directly with the Keplerian velocity components, it is perhaps more useful to determine the distance between the vehicles which actually remains at the time τ^*. This "miss distance" $R(\tau^*)$ is determined by using the approximate velocity components and Keplerian analysis. The results of this process, shown for several times τ^* in Fig. 10, indicate that the miss distance is less than 0.5 miles for τ^* up to $0.8\ \pi$. For comparison purposes, Fig. 10 shows a second curve labeled $R_L(\tau^*)$. This is the miss distance that would occur if the approximate components of velocity were based on neglecting the quadratic terms in the coordinates in Eqs. (10), which is equivalent to retaining only those terms in Eqs. (23–30) which are linear in the relative velocities and coordinates.

Case VI is the same as case V except that the interceptor speed exceeds the target speed by 250 fps, instead of 100 fps. Figure 11 shows miss distances for this case based on both quadratic theory and linear theory. In addition, for the path intended to rendezvous at $\tau^* = 0.7\ \pi$, the minimum distance between the vehicles is shown for comparison.

Figures 10 and 11 indicate that, for the two cases studies, the quadratic theory is considerably better than the linear theory for purposes of determining velocity components required for rendezvous.

IV. Conclusions

An approximate analytical solution has been obtained for the equations that describe the motion of an interceptor relative to a target vehicle in a nearly circular orbit. Although the eccentricity of the target orbit is restricted to small values, the results of this investigation are directly applicable to a large group of near-earth target orbits. When the target orbit is circular, and only linear terms in the coordinates are retained, the solution yields the results of Clohessy-Wiltshire, whereas if quadratic terms are retained, London's results are obtained except for discrepancies in two terms in his solution.

Numerical results indicate that the new solution is valid not only for the terminal phase of the rendezvous maneuver but also that it furnishes accurate results when the relative distance is quite large, e.g., on the order of a thousand miles. In the cases investigated, the linearized solution is applicable when the rendezvous time interval is small, but the quadratic terms in the distance must be retained when the distance becomes large and the rendezvous time corresponds to one or two periods of the target orbit.

An approximate explicit expression is derived for the velocity components required to effect rendezvous at a prescribed time. The accuracy of these results is found to be very high for the two cases investigated.

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